A Modification of the Kynch Theory of Sedimentation

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Introduction

The fundamentals and applications of thickening have been extensively studied, starting with the work of Mishler (1912) and Coe and Clevenger (1916). The objective of those pioneering works was to present a method to calculate the area of a continuous thickener from batch sedimentation experiments. It is significant that up to the present day, the method developed by these workers remains one of the most popular design procedures.

Kynch (1952) presented the first theory of sedimentation. In spite of the fact that the Kynch theory is not applicable to problems currently encountered in industrial thickening, the design procedure based on it, called the Kynch method of thickener design and put forward by Talmage and Fitch (1955), has been used extensively up to now. This procedure frequently gives different results from the more reliable method of Coe and Clevenger. The problem arises from the fact that industrial slurries are always compressible to a greater or lesser degree and the Kynch method does not take into account compressibility.

The simplicity of the Kynch method of thickener design has induced research workers in the field to find a similar method, one based on the Kynch theory but that would take the discrepancy between the theory and the experimental results for industrial slurries into account. Two recent examples are papers by Tiller (1981) and Fitch (1983) in which the authors propose an alternative graphical procedure to obtain the suspension concentration and the corresponding settling velocity.

In this work we present a modification of the Kynch theory of sedimentation that permits the description of the settling of flocculated solid-fluid suspensions.

Kynch Theory of Sedimentation

In 1952 Kynch developed the first theory of batch sedimentation. This theory is purely kinematic and describes the sedimentation under gravity of solid particles through a fluid as a wave propagation phenomenon. Kynch established that in such a process discontinuities may appear as shock waves or contact discontinuities. He assumed that the velocity of the solid particles in a fluid depended only on the local concentration. No wall effect was taken into account, so that the motion is one-dimensional and the particles are considered to be of the same size and shape.

Let $\phi(x, t)$ be the concentration of the solid component as volumetric fraction and v_s its velocity; then the continuity equation for the solid may be written in the form:

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi v_s) = 0, \quad v_s < 0, \tag{1}$$

in $\Omega = \{(z, t) | z \leq L, 0 \leq t\}$, where L is the initial height of the mixture. Let $f(\phi) = v_s\phi$ be the solid flux density; then Eq. 1 becomes:

$$\frac{\partial \phi}{\partial t} + \frac{\partial f(\phi)}{\partial z} = 0, \tag{2}$$

in Ω.

Along lines of discontinuities $z = z(t)$, the concentration satisfies the jump mass balance, also known as the Rankine-Hugoniot condition:

$$\delta(\phi^+ - \phi^-) = f(\phi^+) - f(\phi^-), \tag{3}$$

where $\delta = \dot{z}(t)$ is the speed of propagation of the discontinuity, and $\phi^\pm = \phi\{z(t) \pm 0, t\}$.

Kynch assumed that near the point $z = 0$ there is a continuous but extremely rapid increase in concentration from the initial concentration ϕ_o to the maximum possible concentration ϕ_∞. Therefore the initial and boundary conditions were assumed to be:

$$\phi(z, 0) = \phi_o, \quad 0 \leq z \leq L, \tag{4}$$

$$\phi(0, t) = \phi_\infty, \quad 0 < t. \tag{5}$$

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Kynch's theory describes quite well the sedimentation of incompressible solid particles such as glass beads and the settling of nonflocculated mineral particles. In general, the sedimentation of flocculated suspensions, such as those commonly encountered in the mineral industry, cannot be described with this theory, since the consolidation of thick slurries involves the presence of forces not taken into account by a kinematic model.

Two recent articles (Tiller, 1981; Fitch, 1983) extend the validity of the Kynch theory to compressible suspensions. Both are based on the analysis of the settling curve and present a graphical construction to obtain the settling variables (ϕ, v_s). Without diminishing the practical value of this approach, it must be pointed out that the extension in both cases violates the fundamentals of the theory. If the Kynch theory is to remain the basis of the model, concentration and shock waves should have a constant rate of propagation, that is, the characteristics must be straight lines.

Modified Kynch Model

We will assume that for real slurries all conditions stipulated by Kynch are valid, except that we restrict their validity to solid concentrations less than or equal to a critical concentration ϕ_c. This is the highest concentration that can be obtained by initial settling tests.

The boundary condition in Eq. 5 cannot be applied now because the sedimented particles at concentration ϕ_c keep a certain quantity of water that can only be expelled from the sediment by means of compression. The weight of the sedimented solid compresses the inferior layers, increasing their concentration. To take into account this phenomenon we propose to modify the boundary condition used by Kynch.

In batch sedimentation, the velocity of the solid particles and the velocity of the fluid at the bottom of the container ($z = 0$) are zero. Therefore, neglecting inertia forces, a momentum balance applied to the solid constituent of the pulp leads to:

$$\frac{\partial p_s}{\partial z} = -\Delta\rho\phi g, \tag{6}$$

at $z = 0$, where p_s is the solid pressure, (a normal stress transmitted through solid-solid contacts), $\Delta\rho$ is the solid-fluid density difference, and g is the acceleration of gravity constant. The hydrodynamic force is also zero at $z = 0$ since it is a function of the solid-fluid relative velocity. Becker (1982) and Bustos (1984), assuming that the solid pressure is a function of concentration only, use the following constitutive equation for p_s:

$$p_s = ae^{b\phi}. \tag{7}$$

Then, from Eq. 6 we deduce that:

$$\frac{\partial\phi}{\partial z} = -\frac{\Delta\rho\phi g}{\partial p_s/\partial\phi} = g_1(0, t), \quad 0 < t, \tag{8}$$

at $z = 0$. This equation is the new boundary condition that we propose to replace Eq. 5 in the Kynch model.

Numerical Solution

Solutions to the Kynch problem are discontinuous in general. Therefore we define weak solutions as a real valued function $\phi(z, t)$, bounded and measurable, such that the following relationship holds:

$$\int_o^\infty \int_o^L \left[\phi \frac{\partial w}{\partial t} + f(\phi)\frac{\partial w}{\partial z}\right] dz\, dt + \int_o^\infty w(z, 0)\phi_o(z)\, dz - \int_o^\infty w(0, t)\phi(0, t)\, dt = 0, \tag{9}$$

for any three-times-differentiable test function $w(z, t)$ that is zero outside of a closed and bounded domain. Here the boundary condition $\phi(0, t)$ is obtained by replacing Eq. 8 in the Kynch equation at $z = 0$ and then integrating to obtain:

$$\phi(0, t) = \int_o^t f'[\phi(0, \tau)]g_1(0, \tau)\, d\tau + \phi_c, \quad 0 < t \tag{10}$$

The advantage of the weak formulation of Kynch's problem, lies in the fact that the solution $\phi(x, t)$ of Eq. 9 satisfies Eq. 2 at all points of continuity, and along each line of discontinuity $z(t)$ it satisfies the Rankine-Hugoniot condition, Eq. 3 (Oleinik, 1957).

In the class of weak solutions, uniqueness is lost and an additional criterion is needed to single-out the admissible weak solution. One criterion is Lax's entropy condition for shocks (Lax, 1957; Petty, 1975):

$$f'(\phi^-) \geqq \delta \geqq f'(\phi^+), \tag{11}$$

with δ, ϕ^+, ϕ^- as defined in Eq. 3. The physical interpretation of Lax's entropy condition is that the speed of propagation of concentration waves along the characteristic lines in front of the shock must be smaller than the displacement velocity of the shock, which in turn must be smaller than the velocity of the waves in back of it. This interpretation is well known in the theory of compressible fluid mechanics, and it is mentioned in Kynch's paper.

To solve the modified Kynch problem numerically, we use a three-point upwind finite-difference scheme, as described below. This is justified since the numerical solution at any grid point will be influenced more by what is happening below (or upwind) of that point than by what is happening above it. Let the domain $\Omega = \{(z, t) | 0 \leqq z \leqq L, 0 \leqq t\}$ be covered by a uniform rectangular grid defined by the lines $t = nk$, $z = jh$, where h and k are the space and time steps, respectively. The approximating finite-difference scheme to Eq. 2 is:

$$\phi_j^{n+1} = \phi_j^n - \lambda\left(\frac{3}{2}f_j^n - 2f_{j-1}^n + \frac{1}{2}f_{j-2}^n\right); \quad j = 2, \ldots, N \qquad n = 0, 1, \ldots \tag{12}$$

for $f'(\phi_j^n) > 0$. Hence $\lambda = k/h$ and N is equal to the initial height L divided by h. For the first space step we use the Lax-Friedrich scheme (Bustos 1984):

$$\phi_1^{n+1} = \frac{1}{2}(\phi_0^n + \phi_2^n) - \frac{\lambda}{2}(f_2^n - f_0^n), \quad n = 0, 1, \ldots \tag{13}$$

We are using the standard notation where ϕ_j^n is the value of the mesh function at the point (jh, nk) and $f_j^n = f(\phi_j^n)$.

We can write Eqs. 12 and 13 in "conservation form" (Lax and Wendroff, 1960):

$$\phi_j^{n+1} = \phi_j^n - \lambda[q(\phi_j^n, \phi_{j-1}^n) - q(\phi_{j-1}^n, \phi_{j-2}^n)], \qquad (14)$$

for all j and n, where:

$$q(\phi_j^n, \phi_{j-1}^n) = \frac{3}{2}f_j^n - \frac{1}{2}f_{j-1}^n, \quad j = 2, \ldots, N \qquad (15)$$

$$q(\phi_1^n, \phi_0^n) = \frac{1}{2}(f_1^n + f_0^n) - \frac{1}{2\lambda}(\phi_1^n - \phi_0^n), \qquad (16)$$

satisfy the consistency condition

$$q(\phi, \phi) = f(\phi) \qquad (17)$$

In general, at least two shocks develop, one that moves up from the bottom of the slurry and a second one, corresponding to the water-suspension interface $I(t)$, that goes down. Since the finite-difference scheme can be written in conservation form, we know from a theorem by Lax and Wendroff that the limit of the approximate solution is a weak solution. This implies that the numerical solution automatically satisfies the jump condition across a discontinuity. In each time step we calculate the mesh function from below, taking into account the first shock only. To determine the second shock, we use the fact that the total mass of the solid constituent in the mixture is constant:

$$\int_0^{I(t)} \phi(z, t)\, dz = \int_0^L \phi_o\, dz. \qquad (18)$$

The existence and uniqueness of the solution to the modified Kynch model are discussed thoroughly in Bustos (1984).

Simulation of Compressible Suspensions

To simulate a compressible suspension by the modified Kynch model, we make use of experimental results on flocculated tailings from the copper concentrator of the Chuquicamata Division of Codelco Chile. The material has a density of 2,500 kg/m^3.

From standard sedimentation and consolidation tests, Becker (1982) obtained the following parameters for the flux function $f(\phi)$ and the solid pressure p_s:

$$f(\phi) = u_\infty \phi(1 - \phi)^{n+1}, \quad 0 \leq \phi < \phi_c,$$
$$p_s(\phi) = ae^{b\phi}, \quad \phi_c < \phi \leq \phi_\infty, \qquad (19)$$

with

$$u_\infty = -6.05 \times 10^{-4}\ \text{m/s}$$
$$n = 11.59$$
$$a = 5.35\ \text{N/m}^2$$
$$b = 17.9$$

Discussion

The principal discrepancy between Kynch's theory and the settling behavior of any flocculated suspension is observed beyond the compression point, i.e., after the solid concentration that is immediately below the water-suspension interface reaches its critical value ϕ_c. In Kynch's theory, settling finishes at this point, and $\phi_c = \phi_\infty$, whereas in practice sedimentation may continue for a much longer time.

Strictly speaking, the Kynch model for flocculated suspensions is valid only for $\phi \leq \phi_c$, since after that concentration the solid flux is not a function of the concentration only and Eq. 19 is no longer valid. In a more general sense, the Kynch theory (field equation) is still valid, but the constitutive equation for the solid flux, as given by Eq. 19 or any other such relationship, is not. There is no need to postulate other constitutive equations for the solid flux at the concentration range $\phi_c < \phi \leq \phi_\infty$, since a knowledge of the concentration at all points $(0, t)$ that may be obtained from a boundary condition such as Eq. 8, permits the calculation of concentration and solid flux at all other points of Ω. The starting point to obtain these values of the concentration at $(0, t)$ is that $\phi(0, 0^+) = \phi_c$.

From a mathematical point of view, the end of the settling period in Kynch's theory is characterized by parallel straight-line characteristics leaving from the abcsisa, for $t > 0$, with slopes depending on the slope of the flux curve at the concentration $\phi = \phi_\infty$, as can be seen in Figure 1. It is clear that if we pretend that Kynch's theory includes the consolidation effects, the characteristics for $\phi > \phi_c$ that leave the abcissa at $t > 0$ should be straight lines with decreasing slopes from $f'(\phi_c)$ to $f'(\phi_\infty)$. This is precisely what can be observed in Figure 2. It is interesting to keep in mind that these characteristic lines were obtained by point-to-point numerical calculation and were not drawn merely by intuition.

In spite of the fact that we do not have sufficient experimental data to establish the applicability of this modification of Kynch's theory, one can expect that the theory should reasonably represent the sedimentation of flocculated suspensions with low compressibility. In these cases one can anticipate that for $\phi > \phi_c$, the characteristic lines will approximate straight lines. If the compressibility of the suspension is large, the characteristics will be curved lines. Here it is possible that the water-suspension interface is still well represented, but not the height-concentration curves.

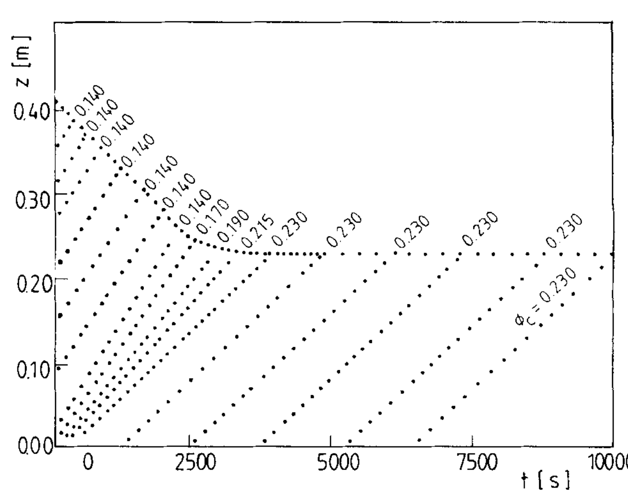

Figure 1. Simulated z versus t plot for the settling of a compressible suspension. Unmodified Kynch theory with $\phi_c = \phi_\infty$.

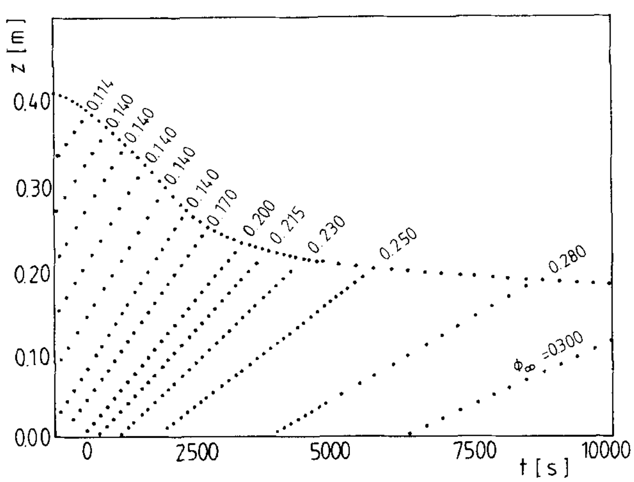

Figure 2. Simulated *z* versus *t* plot for the settling of a compressible suspension. Modified Kynch theory.

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Notation

a = parameter, Eq. 7
b = parameter, Eq. 7
e = base of natural logarithm
f = solid flux density or solid volumetric velocity
g = acceleration of gravity constant
g_1 = function, Eq. 8
h = space step
$I(t)$ = water-suspension interface
k = time step
L = initial height of mixture
$N = L/h$
n = exponent in Richardson and Zaki Eq. 19
p_s = solid partial pressure
q_j = functions, Eqs. 15, 16
t = time
u_∞ = terminal settling velocity
v_s = velocity of solid constituent of mixture
w = test function, Eq. 9
z = vertical space variable

Greek letters

Ω = domain
δ = speed of propagation of a discontinuity
ϕ = concentration expressed in volumetric fraction of solids
ϕ_c = critical concentration
ϕ_∞ = final concentration in Kynch model
ρ_s = density of solid constituent
ρ_f = density of fluid constituent
$\Delta\rho$ = solid-fluid density difference
$\lambda = h/k$

Subscripts

j = index for space variable
o = initial concentration
s = solid

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